



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

On Rotations in Space of Four Dimensions.

BY F. N. COLE.

1.

LINEAR CONFIGURATION IN FOUR-DIMENSIONAL SPACE.

1. In a four-dimensional space of constant zero curvature, suppose any point to be selected, and through this point four linear solids mutually at right angles to each other, to be drawn. Taken in pairs, these four solids intersect in six planes, and taken by threes they intersect in four straight lines. Taken all together, they have only the selected point in common. This system of the point, the four lines, the six planes, and the four solids may be employed as a coordinate configuration. The point will be the origin, the four lines may be called the axes of x , y , z and w , the six planes the planes of xy , xz , xw , yz , yw and zw , the four linear solids, the solids of xyz , xyw , xzw and yzw respectively. The coordinate solids, in the order as written, are defined by the equations

$$w = 0, z = 0, y = 0, x = 0;$$

the planes by

$$w = 0, w = 0, z = 0, x = 0, x = 0, x = 0,$$

$$z = 0, y = 0, y = 0, w = 0, z = 0, y = 0,$$

and the axes by

$$y = 0, x = 0, x = 0, x = 0,$$

$$z = 0, z = 0, y = 0, y = 0,$$

$$w = 0, w = 0, w = 0, z = 0.$$

In pairs, the four axes determine the six coordinate planes; in threes, they determine the four coordinate solids. The six planes taken in pairs would seem at first sight to intersect in fifteen straight lines. But the two planes xy and zw have evidently no element in common except the origin, since this is the only point for which we can have simultaneously $w = 0, z = 0, x = 0, y = 0$. The same is evidently true for any pair of planes whose symbols have no letter in common. There are three such pairs, and accordingly the fifteen apparent lines of intersection reduce to twelve. Again, the pairs of planes xy and xz , xy and

yz , xz and xy , all intersect on the line $x = 0$, $y = 0$, $z = 0$. All the pairs of planes can be similarly arranged, in four sets of three each, which have the same line of intersection. The twelve lines therefore reduce to the four coordinate axes, each counted three times.

Of the thirty combinations of the coordinate planes by threes, four sets which have each a common line have just been considered. Beside these, the four sets in each of which the same letter occurs three times in the symbols for the three planes, determine each a solid corresponding to the repeated letter. Thus the planes xy , xz and xw determine the solid $x = 0$. The remaining twenty-two sets of three planes determine no elementary configuration.

The numerical arrangement of the parts of the coordinate configuration may therefore be briefly expressed as follows:

Each coordinate solid contains three coordinate planes and three coordinate axes. Each coordinate plane is contained in two coordinate solids and contains two coordinate axes. Each coordinate axis is contained in three coordinate solids and in three coordinate planes.

The coordinate (rectangular) configurations of the ordinary plane and space geometries divide the angular space about the origin into four plane right angles and eight right trihedral angles respectively.

Similarly in space of four dimensions the angular space about the origin is divided by the coordinate configuration just discussed into sixteen right tetrahedral angles.

2. A point in four-dimensional space is determined by four coordinates referred to the four coordinate solids. An equation of the first degree between these four coordinates defines a linear three-dimensional configuration. Such a configuration I shall call a *lineoid*. Among the lineoids the coordinate solids are of course included.

Geometrically a lineoid is determined by four points. For although four points are determined by sixteen constants, each point has, while remaining within the lineoid, still three degrees of freedom. Of the sixteen constants there remain therefore only $16 - 12$, or 4, and this number corresponds to the four essential constants in the equation of the lineoid.

Two equations between the four coordinates analytically define a plane. But if two lineoids intersect in a given plane, any pair of linear combination of the two lineoids will intersect in the same plane. Two such pairs of linear combinations involve two essential constants. These subtracted from the eight

constants which determine the two lineoids leave six constants which determine their plane of intersection. Geometrically a plane is determined by three points. For each of the three points has, while remaining in the plane, two degrees of freedom, so that we have $3 \times 2 = 6$ essential constants.

Similarly three linear equations or two points determine a straight line. The two points involve eight constants, but as each has one degree of freedom on the line, these eight reduce to six essential constants.

The linear configurations in four-dimensional space are therefore determined as follows:

A lineoid by 4 constants.
 A plane by 6 constants.
 A line by 6 constants.
 A point by 4 constants.

The dualistic relation between the lineoids and points on the one hand and the planes and lines on the other appears clearly in this table.

3. Among these linear configurations we are especially interested in those which contain the origin, and for which accordingly the constant terms in all the defining equations disappear. In this case a lineoid is determined by three further conditions, a plane by four, and a line by three. The geometry of these configurations is therefore analogous in form to that of the planes, lines and points of ordinary three-dimensional space.

As, however, the greater portion of the following developments hold for all linear configurations, whether they include the origin or not, I shall state them for the most part in their full generality.

If the equation of a lineoid be

$$ax + by + cz + dw + e = 0,$$

then we may write $\frac{a}{\sqrt{a^2 + b^2 + c^2 + d^2}} = \cos \alpha$, $\frac{b}{\sqrt{a^2 + b^2 + c^2 + d^2}} = \cos \beta$, $\frac{c}{\sqrt{a^2 + b^2 + c^2 + d^2}} = \cos \gamma$, $\frac{d}{\sqrt{a^2 + b^2 + c^2 + d^2}} = \cos \delta$, where the four cosines are connected by the identity $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = 1$.

These four cosines I call the direction cosines of the corresponding lineoid. The quantity $\frac{e}{\sqrt{a^2 + b^2 + c^2 + d^2}}$ may be called the perpendicular distance from the origin on the given lineoid.

If the direction cosines of two lineoids be

$$\begin{aligned} &\cos \alpha, \cos \beta, \cos \gamma, \cos \delta, \\ &\cos \alpha', \cos \beta', \cos \gamma', \cos \delta' \end{aligned}$$

respectively, we may form from these the six determinants of the second order:

$$\begin{array}{ll} \cos \alpha \cos \beta' - \cos \beta \cos \alpha' & \cos \beta \cos \gamma' - \cos \gamma \cos \beta', \\ \cos \alpha \cos \gamma' - \cos \gamma \cos \alpha' & \cos \beta \cos \delta' - \cos \delta \cos \beta', \\ \cos \alpha \cos \delta' - \cos \delta \cos \alpha' & \cos \gamma \cos \delta - \cos \delta \cos \gamma'. \end{array}$$

The six quantities obtained by dividing each of these by the square root of the sum of the square of all of them I denote by P_{12} , P_{13} , P_{14} , P_{23} , P_{24} , P_{34} , and these I call the six direction cosines of the given planes. Thus

$$P_{12} = \frac{\cos \alpha \cos \beta' - \cos \beta \cos \alpha'}{\sqrt{1 - (\cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma' + \cos \delta \cos \delta')^2}}.$$

The denominator of the P 's can only vanish when $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, $\delta = \delta'$, in which case the two given lineoids coincide in all but their constant terms.

Beside the 6 P 's the plane has also for determining elements the perpendiculars from the origin on the two given lineoids.

Between the P 's there is the identity

$$P_{12}^2 + P_{13}^2 + P_{14}^2 + P_{23}^2 + P_{24}^2 + P_{34}^2 = 1. \quad (1)$$

But since a plane through the origin* is determined by four constants, there must be still another identity connecting the P 's. This is

$$P_{12}P_{34} + P_{13}P_{42} + P_{14}P_{23} = 0 \quad (2)$$

as is readily seen from the development of the identically vanishing determinant

$$\begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma & \cos \delta \\ \cos \alpha' & \cos \beta' & \cos \gamma' & \cos \delta' \\ \cos \alpha & \cos \beta & \cos \gamma & \cos \delta \\ \cos \alpha' & \cos \beta' & \cos \gamma' & \cos \delta' \end{vmatrix}.$$

4. If we multiply the 6 P 's by an arbitrary quantity K , we may regard the resulting quantities as the 6 *homogeneous* coordinates of a plane through the origin. These six coordinates are then identical with the six Plücker coordinates of a line in three-dimensional space. The geometry of planes through the origin in four-dimensional space is therefore exactly parallel to the Plücker line geometry, and every proposition of the one theory can be transferred to the other, so far as the geometrical distinction between the three- and the four-dimensional spaces interposes no obstacle. The examination of this correspondence in detail I intend to treat in a future paper. For the present I will merely estab-

*In treating the direction cosines alone we may, of course, suppose the given plane to pass through the origin.

lish two fundamental propositions for the geometry of the planes of a four-dimensional space.

$$\begin{aligned} 5. \text{ Two lineoids, } & ax + by + cz + dw + e = 0, \\ & a'x + b'y + c'z + d'w + e' = 0 \end{aligned}$$

I call perpendicular to each other, if $aa' + bb' + cc' + dd' = 0$, or in terms of the direction cosines of the two lineoids,

$$\cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma' + \cos \delta \cos \delta' = 0.$$

If we have two pair of lineoids such that both lineoids of the one pair are perpendicular to both of the other pair, the planes of intersection of each pair are naturally called perpendicular to each other. Two such planes will have no line in common. I call two such pairs of planes *absolutely* perpendicular to each other. Thus any pair of coordinate planes whose symbols contain no common letter, as xy and zw , are absolutely perpendicular to each other.

6. We can now establish at once the important proposition: Through a given point in a given plane only one plane can be passed which is absolutely perpendicular to the given plane. That this is the case is indicated by the fact that four conditions are requisite for absolute perpendicularity, and these are sufficient to determine the four essential constants of the second plane. We verify the proposition by the actual determination of the six direction cosines of the second plane in terms of those of the first.

We distinguish the different lineoids by subscripts 1, 2, 3, 4, and suppose that the planes (12) and (34) to be the first and second planes respectively. The four equations of conditions are then

$$\begin{aligned} \cos \alpha_1 \cos \alpha_3 + \cos \beta_1 \cos \beta_3 + \cos \gamma_1 \cos \gamma_3 + \cos \delta_1 \cos \delta_3 &= 0, \\ \cos \alpha_1 \cos \alpha_4 + \cos \beta_1 \cos \beta_4 + \cos \gamma_1 \cos \gamma_4 + \cos \delta_1 \cos \delta_4 &= 0, \\ \cos \alpha_2 \cos \alpha_3 + \cos \beta_2 \cos \beta_3 + \cos \gamma_2 \cos \gamma_3 + \cos \delta_2 \cos \delta_3 &= 0, \\ \cos \alpha_2 \cos \alpha_4 + \cos \beta_2 \cos \beta_4 + \cos \gamma_2 \cos \gamma_4 + \cos \delta_2 \cos \delta_4 &= 0. \end{aligned}$$

From these we readily deduce

$$\begin{aligned} P_{12} \cos \beta_3 + P_{13} \cos \gamma_3 + P_{14} \cos \delta_3 &= 0, \\ P_{12} \cos \beta_4 + P_{13} \cos \gamma_4 + P_{14} \cos \delta_4 &= 0, \\ P_{12} \cos \alpha_3 - P_{23} \cos \gamma_3 - P_{24} \cos \delta_3 &= 0, \\ P_{12} \cos \alpha_4 - P_{23} \cos \gamma_4 - P_{24} \cos \delta_4 &= 0, \\ P_{13} \cos \alpha_3 + P_{23} \cos \beta_3 - P_{34} \cos \delta_3 &= 0, \\ P_{13} \cos \alpha_4 + P_{23} \cos \beta_4 - P_{34} \cos \delta_4 &= 0, \\ P_{14} \cos \alpha_3 + P_{24} \cos \beta_3 + P_{34} \cos \gamma_3 &= 0, \\ P_{14} \cos \alpha_4 + P_{24} \cos \beta_4 + P_{34} \cos \gamma_4 &= 0. \end{aligned}$$

And from these again, if we distinguish the P 's of the second plane by accents, we have

$$\begin{aligned} P_{12}P'_{23} - P_{14}P'_{34} &= 0 & P'_{13} &= -\frac{P_{24}}{P_{12}}P'_{34}, \\ P_{12}P'_{24} + P_{13}P'_{34} &= 0 & P'_{14} &= \frac{P_{23}}{P_{12}}P'_{34}, \\ P_{12}P'_{13} + P_{24}P'_{34} &= 0 & \therefore P'_{23} &= \frac{P_{14}}{P_{12}}P'_{34}, \\ P_{12}P'_{14} - P_{23}P'_{34} &= 0 & P'_{24} &= -\frac{P_{13}}{P_{12}}P'_{34}, \\ P_{13}P'_{12} + P_{34}P'_{24} &= 0 & P'_{12} &= -\frac{P_{34}}{P_{13}}P'_{24} = \frac{P_{34}}{P_{12}}P'_{34}. \end{aligned}$$

If now we write

$$P'_{12} = KP_{34}$$

we have

$$P'_{13} = KP_{42},$$

$$P'_{14} = KP_{23},$$

$$P'_{23} = KP_{14},$$

$$P'_{24} = KP_{31},$$

$$P'_{34} = KP_{12}.$$

But since $\Sigma P_{ik}^2 = 1$ and $\Sigma P'_{ik}^2 = 1$, we have $K^2 = 1$, $K = \pm 1$.

We may take either of the values of K . If we take $K = +1$ we have

$$\begin{aligned} P'_{12} &= P_{34}, \\ P'_{13} &= P_{42}, \\ P'_{14} &= P_{23}, \\ P'_{23} &= P_{14}, \\ P'_{24} &= P_{31}, \\ P'_{34} &= P_{12}. \end{aligned} \tag{3}$$

As a final verification we have

$$P'_{12}P'_{34} + P'_{13}P'_{42} + P'_{14}P'_{23} = P_{34}P_{12} + P_{42}P_{13} + P_{23}P_{14} = 0.$$

The equations (3), regarded as defining the transition from the given plane to its absolute perpendicular plane, are equivalent in the Plücker geometry to the analytic definition of a dualistic transformation. The relation between the planes P and P' is evidently a reciprocal one.

7. These equations may also be simply interpreted within the four-dimensional space as follows: We have regarded a plane as the intersection of two lineoids. We may also regard it as determined by two straight lines. For convenience we will suppose the plane and its determining elements all to pass through the

origin. If the coordinates of any point on any one of the determining lines be x, y, z, w , we may call the quantities

$$\frac{x}{\sqrt{x^2+y^2+z^2+w^2}}, \quad \frac{y}{\sqrt{x^2+y^2+z^2+w^2}}, \quad \frac{z}{\sqrt{x^2+y^2+z^2+w^2}}, \quad \frac{w}{\sqrt{x^2+y^2+z^2+w^2}}$$

the direction cosines of the line. From the direction cosines of the two determining lines we may then form determinants P' of the second order as before in the case of the lineoids.

If a line and a lineoid have the same direction cosines they may be called perpendicular to each other. It appears at once that if a plane be determined by two lineoids and a second plane by two lines whose direction cosines are respectively equal to those of the lineoid, these two planes are absolutely perpendicular to each other. If, therefore, the direction cosines of a plane as determined by two lineoids be P_{ik} and those of the same plane as determined by two lines be P'_{ik} , the equations (3) hold between these two sets of coordinates. In other words, the equations (3) define the transformation from a lineoid geometry to a line geometry.

These results can of course be easily verified analytically by expressing the direction cosines of a line in terms of the direction cosines of the three lineoids which intersect in the line.

8. The condition that two planes shall have a common line is also readily obtained. Thus, if the two planes be determined as before as the intersection of the lineoids 1 and 2, 3 and 4, each of these lineoids must contain the common line of the two planes. The condition for this is obviously

$$\begin{vmatrix} \cos \alpha_1, & \cos \beta_1, & \cos \gamma_1, & \cos \delta_1 \\ \cos \alpha_2, & \cos \beta_2, & \cos \gamma_2, & \cos \delta_2 \\ \cos \alpha_3, & \cos \beta_3, & \cos \gamma_3, & \cos \delta_3 \\ \cos \alpha_4, & \cos \beta_4, & \cos \gamma_4, & \cos \delta_4 \end{vmatrix} = 0.$$

Expanding this determinant in quadratic minors, we have at once

$$P_{12}P'_{34} + P_{13}P'_{42} + P_{14}P'_{23} + P_{23}P'_{14} + P_{42}P'_{13} + P_{34}P'_{12} = 0. \quad (4)$$

This is identical with the Plücker condition that two straight lines in ordinary space shall have a common point.

9. If a plane P' have a line in common with a given plane P , and also with the plane absolutely perpendicular to P , we have the two equations of condition

$$\begin{aligned} P_{12}P'_{34} + P_{13}P'_{42} + P_{14}P'_{23} + P_{23}P'_{14} + P_{42}P'_{13} + P_{34}P'_{12} &= 0, \\ P_{12}P'_{12} + P_{13}P'_{13} + P_{14}P'_{14} + P_{23}P'_{23} + P_{24}P'_{24} + P_{34}P'_{34} &= 0. \end{aligned}$$

From the symmetry of these equations it is at once evident the plane P has a line in common not only with P' but also with the plane absolutely perpendicular to P' . The relation between the two planes is therefore a reciprocal one. Two such planes I call *simply* perpendicular to each other. There are, therefore, ∞^2 planes simply perpendicular to a given plane. The situation of these planes can be readily understood by the aid of the following consideration: Through the point of intersection of the planes P and P' ∞^1 straight lines can be drawn in each plane. Any two of these lines lying one in the plane P , the other in the plane P' , determine one of the simply perpendicular planes. Through each of the lines in P there, therefore, pass ∞^1 of the simple perpendicular planes, and these cut the plane P' in the bundle of rays through the intersection P and P' . It appears, therefore, that there are ∞^1 planes simply perpendicular to a given plane and cutting it in a given line. For example, the planes xz and yz , or any one of their ∞^1 linear combinations, are simply perpendicular to the plane xy .

In the development of this part of the subject I have not attempted anything like an exhaustive treatment. I have simply aimed to establish systematically so much of the theory of the linear configurations containing the origin in space of four dimensions as is of immediate use in the theory of rotations.

2.

THE GENERAL THEORY OF ROTATION IN FOUR-DIMENSIONAL SPACE.

1. The general collineation in space of four dimensions is defined by the four equations

$$\begin{aligned} x' &= \frac{a_1x + b_1y + c_1z + d_1w + e_1}{a_5x + b_5y + c_5z + d_5w + e_5}, \\ y' &= \frac{a_2x + b_2y + c_2z + d_2w + e_2}{a_5x + b_5y + c_5z + d_5w + e_5}, \\ z' &= \frac{a_3x + b_3y + c_3z + d_3w + e_3}{a_5x + b_5y + c_5z + d_5w + e_5}, \\ w' &= \frac{a_4x + b_4y + c_4z + d_4w + e_4}{a_5x + b_5y + c_5z + d_5w + e_5}, \end{aligned} \tag{1}$$

These involve twenty-four essential constants; that is, there are ∞^{24} possible collineations. That these form a group is clear. We are interested in the subgroup which converts a solid of the second order, more particularly a solid sphere, into itself. The equation of such a surface contains fourteen essential constants. Since these are to remain unchanged, we have fourteen equations

of condition among the twenty-four constants of the general collineation. The desired subgroup contains, therefore, ∞^{10} distinct operations.

This subgroup contains again a further subgroup composed of the rotations of the sphere. As these rotations I define those collineations of the four-dimensional space which not only convert the sphere into itself but also leave its center, assumed to be at the origin, and with it its polar solid plane, the lineoid at infinity, unchanged.* From this definition it appears at once that the denominator in the equations (1) reduce to a single constant term which may be regarded as combined with the coefficients of the numerators, and that the constant terms in the numerators reduce to zero.

The equations for the subgroup of these rotations therefore are of the form

$$\begin{aligned} x' &= a_1x + b_1y + c_1z + d_1w, \\ y' &= a_2x + b_2y + c_2z + d_2w, \\ z' &= a_3x + b_3y + c_3z + d_3w, \\ w' &= a_4x + b_4y + c_4z + d_4w. \end{aligned} \tag{2}$$

If the equation of the invariant sphere be $x^2 + y^2 + z^2 + w^2 = 1$, the coefficients a, b, c, d are further connected by the ten equations of condition

$$\begin{aligned} a_1^2 + a_2^2 + a_3^2 + a_4^2 &= 1, & a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 &= 0, \\ b_1^2 + b_2^2 + b_3^2 + b_4^2 &= 1, & a_1a_3 + b_1b_3 + c_1c_3 + d_1d_3 &= 0, \\ c_1^2 + c_2^2 + c_3^2 + c_4^2 &= 1, & a_1a_4 + b_1b_4 + c_1c_4 + d_1d_4 &= 0, \\ d_1^2 + d_2^2 + d_3^2 + d_4^2 &= 1, & a_2a_3 + b_2b_3 + c_2c_3 + d_2d_3 &= 0, \\ & & a_2a_4 + b_2b_4 + c_2c_4 + d_2d_4 &= 0, \\ & & a_3a_4 + b_3b_4 + c_3c_4 + d_3d_4 &= 0. \end{aligned} \tag{3}$$

The equation (2) contains sixteen constants, and as these are connected by the ten relations (3), it appears that the group of rotation of a four-dimensional space about any fixed point contains ∞^6 distinct operations.

2. The theory of orthogonal transformation has been extensively studied by Cayley,† who has given a general method of expressing the n^2 coefficients of such a transformation in terms of the $\frac{1}{2}n(n-1)$ independent constants of the transformation. In the case of a four-dimensional space, if we call the six independent constants a, b, c, f, g, h , we have

*See also §9.

†Crelle XXXII.

$$\begin{aligned}
Ba_1 &= 1 - \mathfrak{S}^2 + f^2 - a^2 + g^2 - b^2 + h^2 - c^2, \\
Ba_2 &= 2(-a - f\mathfrak{S} + cg - bh), \\
Ba_3 &= 2(-b - cf - \mathfrak{S}g + ah), \\
Ba_4 &= 2(-c + bf - ag - h\mathfrak{S}), \\
Bc_1 &= 2(b + g\mathfrak{S} - cf + ah), \\
Bc_2 &= 2(h + fg + c\mathfrak{S} - ab), \\
Bc_3 &= 1 - \mathfrak{S}^2 + g^2 - b^2 + c^2 - h^2 + a^2 - b^2, \\
Bc_4 &= 2(-f + gh - bc - a\mathfrak{S}), \\
Bb_1 &= 2(a + f\mathfrak{S} - bh + cg), \\
Bb_2 &= 1 - \mathfrak{S}^2 + f^2 - a^2 + b^2 - g^2 + c^2 - h^2, \\
Bb_3 &= 2(-h + fg - ab - c\mathfrak{S}), \\
Bb_4 &= 2(g + fh + b\mathfrak{S} - ac), \\
Bd_1 &= 2(c + h\mathfrak{S} - ag + bf), \\
Bd_2 &= 2(-g + hf - ac - b\mathfrak{S}), \\
Bd_3 &= 2(f + gh + a\mathfrak{S} - bc), \\
Bd_4 &= 1 - \mathfrak{S}^2 + h^2 - c^2 + a^2 - f^2 + b^2 - g^2.
\end{aligned} \tag{4}$$

where $\mathfrak{S} = af + bg + ch$

and $B = 1 + a^2 + b^2 + c^2 + g^2 + f^2 + h^2 + \mathfrak{S}^2$.

The question now arises, and this leads to developments of fundamental importance, whether a rotation in four-dimensional space as defined by the equation (2) and (3) or (4) has any points other than the origin fixed, as is the case in the corresponding problem in three-dimensional space. If there be such points, they will be determined by putting in equations (3) x', y', z', w' , equal respectively to x, y, z, w and solving the resulting equation,

$$\begin{aligned}
(a_1 - 1)x + b_1y + c_1z + d_1w &= 0, \\
a_2x + (b_2 - 1)y + c_2z + d_2w &= 0, \\
a_3x + b_3y + (c_3 - 1)z + d_3w &= 0, \\
a_4x + b_4y + c_4z + (d_4 - 1)w &= 0.
\end{aligned} \tag{5}$$

If these equations have a common solution other than 0, 0, 0, 0, we must have

$$\begin{vmatrix}
a_1 - 1 & b_1 & c_1 & d_1 \\
a_2 & b_2 - 1 & c_2 & d_2 \\
a_3 & b_3 & c_3 - 1 & d_3 \\
a_4 & b_4 & c_4 & d_4 - 1
\end{vmatrix} \equiv 0.$$

Substituting now for the a, b, c, d their values as given by equation (4), we find that this determinant is *not* identically 0, but reduces to

$$\frac{\mathfrak{S}^2}{B}.$$

In order, therefore, that a rotation may have any point other than the origin fixed, we must have

$$\mathfrak{S} = 0.$$

We must, therefore, divide the rotations of a four-dimensional space about a fixed point into two classes, one containing ∞^5 distinct operations, each of which leaves other points beside the origin fixed, and the other containing the remaining ∞^6 distinct operations which do not possess this property.

If \mathfrak{S} be 0, the four equations (5) apparently reduce to three independent ones, which accordingly determine a fixed line. But this is not all. The equations (5) in this case really reduce to two, which accordingly determine a fixed plane. That this is so is readily seen if we write in the four equations in the place of the a, b, c, d their value as given in equation (4), in which we are now to put $\mathfrak{S} = 0$.

We have then

$$\begin{aligned} (-a^2 - b^2 - c^2)x + (a - bh + cg)y + (b - cf + ah)z + (c - ag + bf)w &= 0, \\ (-a + cg + bh)x + (-a^2 - g^2 - h^2)y + (h + fg - ab)z + (-g + hf - ac)w &= 0, \\ (-b - cf + ah)x + (-h + fg - ab)y + (-b^2 - f^2 - h^2)z + (f + gh - bc)w &= 0, \\ (-c + bf - ag)x + (g + fh - ac)y + (-f + gh - bc)z + (-c^2 - f^2 - g^2)w &= 0. \end{aligned} \quad (6)$$

If, now, we multiply the second, third, and fourth equations by a, b , and c respectively and add the results to the first equation, remembering that $af + bg + ch = 0$, the resulting coefficients all vanish identically. Moreover, if we multiply the second, third, and fourth equations by f, g , and h respectively and add them together, the resulting coefficients again all vanish.

We have, then, this result: *Of the ∞^6 rotations in general defined by equations (2) and (3), only a minor class of ∞^5 rotations leave any point in space except the origin fixed. Each of these ∞^5 rotations leaves an entire plane fixed.*

We shall find, however, that these latter ∞^5 rotations do not constitute a group. The resultant of two such rotations does not in general leave any point except the origin fixed.

4. Those rotations which leave a plane fixed, I shall hereafter call "simple" rotations. The fixed plane for such a rotation, it must be noted, is not only fixed

* See Scott's Determinants, p. 233.

in the sense that it is converted into itself, but it is fixed absolutely, i. e. every point in it is fixed by itself.

The plane absolutely perpendicular to the fixed plane is also evidently converted into itself, but its separate points do not remain fixed. Since the spherical solid is also converted into itself, it follows that the circle of intersection of the absolutely perpendicular plane with the sphere is likewise converted into itself. Again, every collineation of the four-dimensional space is also a collineation of any plane which it may leave fixed. It appears, therefore, that the absolutely perpendicular plane is simply rotated through a certain angle into itself. This angle we will call the angle of the rotation the four-dimensional space considered.

The ∞^2 planes which are simply perpendicular to the fixed plane of the "simple" rotation have a line in common with this plane and a line in common with the absolutely perpendicular plane. Of these two lines, the one in the fixed plane remains in every case fixed, while that in the absolutely perpendicular plane is rotated in that plane about the origin through the angle of the rotation. If among these planes we select those which have a given line of intersection with the absolutely fixed plane, the transformation which these undergo is exactly the same as that of planes through the axis of a rotation in three-dimensional space.

5. Returning, now, to the analytic treatment of the subject, we can at once, in case $\mathfrak{S} = 0$, find an interpretation for the six independent constants a, b, c, f, g, h . *These are proportional to the six direction cosines of the fixed plane.*

For the fixed plane is determined by any two of the lineoids defined by equation (6), say the first and second. Calculating the direction cosines of the plane of intersection of these lineoids, and remembering again that $af + bg + ch = 0$, we have at once the six quantities $a^2, ab, ac, ah, -ag, af$.

These we may regard as the six homogeneous coordinates of the fixed plane. Between these we have already the Plucker identity $af + bg + ch = 0$. To obtain the six quantities p , we have only to divide these six coordinates by the square root of the sum of their squares. Thus,

$$\begin{aligned} P_{12} &= \frac{a}{B} & P_{23} &= \frac{h}{B}, \\ P_{13} &= \frac{b}{B} & P_{24} &= \frac{-g}{B}, \\ P_{14} &= \frac{c}{B} & P_{34} &= \frac{f}{B} \text{ where } B = \sqrt{1+a^2+b^2+c^2+f^2+g^2+h^2}. \end{aligned}$$

For the absolutely perpendicular plane we have

$$\begin{aligned} P'_{12} &= \frac{f}{B} & P_{23} &= \frac{c}{B}, \\ P'_{13} &= \frac{g}{B} & P_{24} &= \frac{-b}{B}, \\ P'_{14} &= \frac{h}{B} & P_{34} &= \frac{a}{B}. \end{aligned}$$

Since the fixed plane is determined by four independent constants, while the a, b, c, f, g, h , with the identity $af + bg + ch = 0$, constitute a system of five independent quantities, it is clear that the *extent* of the rotation is measured by a single constant; that is, that the rotation is in itself one-dimensional, a result which agrees with the preceding geometrical consideration.

It remains to determine the extent of the rotation; that is, the angle mentioned above. If we write

$$\begin{aligned} a &= \cos \alpha \tan \frac{\phi}{2}, & b &= \cos \beta \tan \frac{\phi}{2}, & c &= \cos \gamma \tan \frac{\phi}{2}, \\ f &= \cos \delta \tan \frac{\phi}{2}, & -g &= \cos \epsilon \tan \frac{\phi}{2}, & h &= \cos \delta \tan \frac{\phi}{2}, \end{aligned}$$

where the six cosines are the six direction cosines of the fixed plane, that is, the six P 's, so that

$$B = \sqrt{1 + \tan^2 \frac{\phi}{2}} = \sec \frac{\phi}{2},$$

the angle ϕ thus defined is the angle of the rotation.

It will be sufficient to prove this in a single case. We will, therefore, assume $b = c = f = g = h = 0$, so that the fixed plane shall be the plane of zw . From equation (4) we have, then, for the equation of the rotation,

$$\begin{aligned} x' &= \frac{1 - a^2}{1 + a^2} x + \frac{2a}{1 + a^2} y & z' &= z, \\ y' &= \frac{-2a}{1 + a^2} x + \frac{1 - a^2}{1 + a^2} y & w' &= w. \end{aligned}$$

Comparing these with the equation for rotation about the origin in the plane xy ,

$$\begin{aligned} x' &= x \cos \phi - y \sin \phi, \\ y' &= x \sin \phi + y \cos \phi, \end{aligned}$$

we have

$$\begin{aligned} \cos \phi &= \frac{1 - a^2}{1 + a^2}, & \sin \phi &= \frac{-2a}{1 + a^2}, \\ \therefore a^2 &= \frac{1 - \cos \phi}{1 + \cos \phi} = \tan^2 \frac{\phi}{2}. \end{aligned}$$

6. I have already mentioned that the combination of two rotations with fixed planes does not in general give a rotation with a fixed plane. The discussion of this question is included in the general theory of the composition of rotation, to which I now proceed. In the following I obtain expressions for the quantities $a, b, c, f, g, h, \mathfrak{S}$ of a resultant rotation in terms of the same quantities for the two component rotations.

If we write the two component rotations in the form

$$\begin{aligned}x'_i &= c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3 + c_{i4}x_4, \\x''_i &= c'_{i1}x'_1 + c'_{i2}x'_2 + c'_{i3}x'_3 + c'_{i4}x'_4,\end{aligned}$$

and suppose the rotations to occur in order as written, we have for the resultant rotation

$$x'''_i = c''_{i1}x_1 + c''_{i2}x_2 + c''_{i3}x_3 + c''_{i4}x_4,$$

where

$$c''_{ij} = c'_{i1}c_{1j} + c'_{i2}c_{2j} + c'_{i3}c_{3j} + c'_{i4}c_{4j}. \quad (\alpha)$$

Turning now to equation (4), we have

$$\begin{aligned}B''(c''_{11} + c''_{22} + c''_{33} + c''_{44}) &= 1 - \mathfrak{S}''^2 - a''^2 - b''^2 - c''^2 + f''^2 + g''^2 + h''^2 \\&\quad + 1 - \mathfrak{S}''^2 - a''^2 + b''^2 + c''^2 + f''^2 - g''^2 - h''^2 \\&\quad + 1 - \mathfrak{S}''^2 + a''^2 + b''^2 + c''^2 - f''^2 + g''^2 - h''^2 \\&\quad + 1 - \mathfrak{S}''^2 + a''^2 + b''^2 - c''^2 - f''^2 - g''^2 + h''^2 \\&= 4(1 - \mathfrak{S}''^2).\end{aligned}$$

By the aid of equation (α) we then have at once

$$\begin{aligned}&\frac{4BB'}{B''}(1 - \mathfrak{S}''^2) \\&= (1 - \mathfrak{S}'^2 - a'^2 - b'^2 - c'^2 + f'^2 + g'^2 + h'^2)(1 - \mathfrak{S}^2 - a^2 - b^2 - c^2 + f^2 + g^2 + h^2) \\&\quad + (1 - \mathfrak{S}'^2 - a'^2 + b'^2 + c'^2 + f'^2 - g'^2 - h'^2)(1 - \mathfrak{S}^2 - a^2 + b^2 + c^2 + f^2 - g^2 - h^2) \\&\quad + (1 - \mathfrak{S}'^2 + a'^2 - b'^2 + c'^2 - f'^2 + g'^2 - h'^2)(1 - \mathfrak{S}^2 + a^2 - b^2 + c^2 - f^2 + g^2 - h^2) \\&\quad + (1 - \mathfrak{S}'^2 + a'^2 + b'^2 - c'^2 - f'^2 - g'^2 + h'^2)(1 - \mathfrak{S}^2 + a^2 + b^2 - c^2 - f^2 - g^2 + h^2) \\&\quad + 4(a' + f'\mathfrak{S}' - b'h' + c'g')(-a - f\mathfrak{S} + cg - bh) \\&\quad + 4(b' + g'\mathfrak{S}' - c'f' + a'h')(-b - cf - g\mathfrak{S} + ah) \\&\quad + 4(c' + h'\mathfrak{S}' - a'g' + b'f')(-c + bf - ag - h\mathfrak{S}) \\&\quad + 4(-a' - f'\mathfrak{S}' + c'g' - b'h')(a + f\mathfrak{S} - bh + cg) \\&\quad + 4(h' + f'g' + c'\mathfrak{S}' - a'b')(-h + fg - ab - c\mathfrak{S}) \\&\quad + 4(-g' + h'f' - a'c' - b'\mathfrak{S}')(g + fh + b\mathfrak{S} - ac) \\&\quad + 4(-b' - c'f' - g'\mathfrak{S}' + a'h')(b + g\mathfrak{S} - cf + ah) \\&\quad + 4(-h' + f'g' - a'b' - c'\mathfrak{S}')(h + fg + c\mathfrak{S} - ab) \\&\quad + 4(f' + g'h' + a'\mathfrak{S}' - b'c')(-f + gh - bc - a\mathfrak{S}) \\&\quad + 4(-c' + b'f' - a'g' - h'\mathfrak{S}')(c + h\mathfrak{S} - ag + bf) \\&\quad + 4(g' + f'h' + b'\mathfrak{S}' - a'c')(-g + hf - ac - b\mathfrak{S}) \\&\quad + 4(-f' + g'h' - b'c' - a'\mathfrak{S}')(f + gh + a\mathfrak{S} - bc).\end{aligned}$$

From this we readily obtain by multiplying out and recombining the results,

$$\frac{BB'}{B''}(1 - \mathfrak{S}'') = (1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh')^2 - (a'f + af' + b'g + bg' + c'h + ch' + \mathfrak{S} + \mathfrak{S}')^2.$$

$$\begin{aligned} \text{Hence } \mathfrak{S}'' &= + \frac{B''}{BB'} (a'f + af' - b'g + bg' + c'h + ch' + \mathfrak{S} + \mathfrak{S}')^2 \\ &\quad - \frac{B''}{BB'} (1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh')^2 \\ &\quad + 1. \end{aligned}$$

Now, if the component rotations each leave a plane fixed, we have $\mathfrak{S} = \mathfrak{S}' = 0$. And if, in addition, these two fixed planes have a line in common, we have also $a'f + af' + b'g + bg' + c'h + ch' = 0$. But in this case the resultant rotation will also leave the line of intersection of the two planes fixed and consequently will leave a whole plane fixed. We have therefore for this case

$$\begin{aligned} \mathfrak{S}'' &= 0, \\ \frac{B''}{BB'} (1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh')^2 &= 1, \\ \therefore B'' &= \frac{BB'}{(1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh')^2} \\ &= \frac{(1 + a^2 + b^2 + c^2 + f^2 + g^2 + h^2 + \mathfrak{S}^2)(1 + a'^2 + b'^2 + c'^2 + f'^2 + g'^2 + h'^2 + \mathfrak{S}'^2)}{(1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh')} \end{aligned}$$

But since the a, b, c, f, g, h are independent variables, only connected in the present case by the three equations of condition

$$\mathfrak{S} = 0, \mathfrak{S}' = 0, a'f + af' + b'g + bg' + c'h + ch' = 0,$$

it is clear that we have always

$$B'' = \frac{BB'}{(1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh')^2}$$

and consequently

$$\begin{aligned} \mathfrak{S}'' &= \frac{B''}{BB'} (a'f + af' + b'g + bg' + c'h + ch' + \mathfrak{S} + \mathfrak{S}')^2 \\ &= \frac{(a'f + af' + b'g + bg' + c'h + ch' + \mathfrak{S} + \mathfrak{S}')^2}{(1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh')^2}, \\ \mathfrak{S} &= \frac{a'f + af' + b'g + bg' + c'h + ch' + \mathfrak{S} + \mathfrak{S}'}{1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh'}. * \end{aligned} \tag{\beta}$$

* In the extraction of the square root the plus sign must be taken, as any simple example will show.

Again, from equation (4) we have

$$\begin{aligned} B''(c_{12}' - c_{21}') &= 4(a'' + f''\mathfrak{S}''), & B''(c_{34}' - c_{43}') &= 4(f'' + a''\mathfrak{S}''), \\ B''(c_{13}' - c_{31}') &= 4(b'' + g''\mathfrak{S}''), & B''(c_{42}' - c_{24}') &= 4(g'' + b''\mathfrak{S}''), \\ B''(c_{14}' - c_{41}') &= 4(c'' + h''\mathfrak{S}''), & B''(c_{23}' - c_{32}') &= 4(h'' + c''\mathfrak{S}''). \end{aligned}$$

From these, by the aid of the equations (α), we obtain

$$\begin{aligned} 4 \frac{BB'}{B''}(a'' + f''\theta'') &= 2(1 - \theta^2 - a^2 - b^2 - c^2 + f'^2 + g'^2 + h'^2)(a + f\mathfrak{S} - bh + cg) \\ &\quad + 2(a' + f'\mathfrak{S} - b'h + c'g)(1 - \theta^2 - a^2 + b^2 + c^2 + f'^2 - g'^2 - h'^2) \\ &\quad + 4(b' + g'\mathfrak{S}' - c'f' + a'h')(-h + fg - ab - c\theta) \\ &\quad + 4(c' + h'\mathfrak{S}' - a'g' + b'f')(g + fh + b\mathfrak{S} - ac) \\ &\quad - 2(-a' - f'\mathfrak{S}' + c'g' - b'h')(1 - \theta^2 - a^2 - b^2 - c^2 + f'^2 + g'^2 + h'^2) \\ &\quad - 2(1 - \mathfrak{S}^2 - a'^2 + b'^2 + c'^2 + f'^2 - g'^2 - h'^2)(-a - f\mathfrak{S} + cg - bh) \\ &\quad - 4(h' + f'g' + c'\mathfrak{S}' - a'b')(-b - cf - g\mathfrak{S} + ah) \\ &\quad - 4(-g' + h'f' - a'c' - b'\mathfrak{S}')(-c + bf - ag - h\mathfrak{S}), \\ 4 \frac{BB'}{B''}(f'' + a''\mathfrak{S}'') &= 4(-c' + b'f' - a'g' - h'\mathfrak{S}')(b + g\mathfrak{S} - cf + ah) \\ &\quad + 4(g' + f'h' + b'\mathfrak{S}' - a'c')(h + fg + c\mathfrak{S} - ab) \\ &\quad + 2(-f' + g'h' - b'c' - a'\mathfrak{S}')(1 - \mathfrak{S}^2 + a^2 - b^2 + c^2 - f'^2 + g'^2 - h'^2) \\ &\quad + 2(1 - \mathfrak{S}^2 + a'^2 + b'^2 - c'^2 - f'^2 - g'^2 - h'^2)(-f + gh - bc - a\theta) \\ &\quad + 4(-b' - c'f' - g'\mathfrak{S}' + a'h')(c + h\mathfrak{S} - ag + bf) \\ &\quad + 4(-h' + f'g' - a'b' - c'\mathfrak{S}')(-g + hf - ac - b\theta) \\ &\quad + 2(1 - \mathfrak{S}^2 + a'^2 - b'^2 + c'^2 - f'^2 + g'^2 - h'^2)(f + gh - ac - b\mathfrak{S}) \\ &\quad + 2(f' + g'h' + a'\mathfrak{S}' - b'c')(1 - \mathfrak{S}^2 + a^2 + b^2 - c^2 - f'^2 - g'^2 + h'^2). \end{aligned}$$

Combining these we obtain, after a series of easy reductions,

$$\begin{aligned} &(a'' + f'')(1 + \mathfrak{S}'')(1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - hh' - gg')^2 \\ &= (1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh' + \mathfrak{S} + \mathfrak{S}' \\ &\quad + a'f + af' + b'g + bg' + c'h + ch')(a + a' + f + f' - (f' + a')\mathfrak{S} - (f + a)\mathfrak{S}' \\ &\quad + bh' - b'h + c'g - cg' + bc' - b'c + gh' - g'h). \end{aligned}$$

But from equation (β)

$$\begin{aligned} 1 + \mathfrak{S}'' &= \frac{1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh' + \mathfrak{S} + \mathfrak{S}' + c'f + cf' + b'g + bg' + c'h + ch'}{1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh'}, \\ \therefore a'' + f'' &= \frac{a + a' + f + f' - (a + f)\mathfrak{S}' - (a' + f')\mathfrak{S} + bh' - b'h + c'g - cg' + bc' - b'c + gh' - g'h}{1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh'}. \end{aligned}$$

Similarly

$$a'' - f'' = \frac{a + a' - f - f' + (a' - f')\mathfrak{S} + (a - f)\mathfrak{S}' + bh' - b'h + c'g - cg' + b'c - bc' + g'h - gh'}{1 - \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh'}$$

Hence

$$\begin{aligned} a'' &= \frac{a + a' - f\mathfrak{S}' - f'\mathfrak{S} + bh' - b'h + c'g - cg'}{1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh'}, \\ f'' &= \frac{f + f' - a'\mathfrak{S} - a\mathfrak{S}' + bc' - b'c + gh' - g'h}{1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh'}. \end{aligned}$$

We can now write down at once all the formulae for the combination of two rotations. If we denote the common denominator $1 + \mathfrak{S}\mathfrak{S}' - aa' - bb' - cc' - ff' - gg' - hh'$ by D , the formulae are

$$\begin{aligned} Da'' &= a + a' - f\mathfrak{S}' - f'\mathfrak{S} + bh' - b'h + c'g - cg', \\ Db'' &= b + b' - g\mathfrak{S}' - g'\mathfrak{S} - cf' - c'f + a'h - ah', \\ Dc'' &= c + c' - h\mathfrak{S}' - h'\mathfrak{S} + ag' - a'g + b'f - bf', \\ Df'' &= f + f' - a\mathfrak{S}' - a'\mathfrak{S} + bc' - b'c + gh' - g'h, \\ Dg'' &= g + g' - b\mathfrak{S}' - b'\mathfrak{S} + ca' - ac' + f'h - fh', \\ Dh'' &= h + h' - c\mathfrak{S}' - c'\mathfrak{S} + ab' - a'b + fg' - f'g, \\ D\mathfrak{S}'' &= \mathfrak{S} + \mathfrak{S}' + a'f + af' + b'g + bg' + c'h + ch'. \end{aligned}$$

The formulae may be tested by aid of the equation

$$\mathfrak{S}'' = a''f'' + b''g'' + c''h''.$$

It will be found that this condition is satisfied.

The condition that the resultant of two simple rotations shall be a simple rotation is now clear. We must have simultaneously $\mathfrak{S}'' = 0$, $\mathfrak{S} = 0$, $\mathfrak{S}' = 0$, which require

$$a'f + af' + b'g + bg' + c'h + ch' = 0.$$

That is, *the resultant of two simple rotations is itself a simple rotation when and only when the fixed planes of the component rotations have a line in common.* That this condition was *sufficient* was already clear from geometrical considerations. It now appears that it is also necessary.

7. I determine next the resultant of two simple rotations whose fixed planes are absolutely perpendicular to each other, the angles of the rotation being ϕ and ϕ' respectively, and K and K' denoting the $\tan \frac{\phi}{2}$ and $\tan \frac{\phi'}{2}$ respectively. If

the six direction cosines of the one plane be $P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}$, those of the other are $P_{34}, P_{42}, P_{23}, P_{14}, P_{31}, P_{12}$. We have then

$$\begin{aligned} a &= KP_{12}, \quad b = KP_{13}, \quad c = KP_{14}, \quad f = KP_{34}, \quad g = -KP_{24}, \quad h = KP_{23}, \\ a' &= K'P_{34}, \quad b' = K'P_{42}, \quad c' = K'P_{23}, \quad f' = K'P_{12}, \quad g' = -K'P_{31}, \quad h' = K'P_{14}. \end{aligned}$$

The equations of combination become in this case

$$\begin{aligned} a'' &= \frac{a + a'}{1 - aa' - bb' - cc' - ff' - gg' - hh'} = \frac{KP_{12} + K'P_{34}}{1 - 2KK'(P_{12}P_{34} + P_{13}P_{42} + P_{14}P_{23})} \\ &= KP_{12} + K'P_{34}, \\ b'' &= KP_{13} + K'P_{42}, \\ c'' &= KP_{14} + K'P_{23}, \\ f'' &= KP_{34} + K'P_{12}, \\ g'' &= KP_{24} + K'P_{31}, \\ h'' &= KP_{23} + K'P_{14}, \\ \mathfrak{S} &= KK'(P_{12}^2 + P_{13}^2 + P_{14}^2 + P_{23}^2 + P_{24}^2 + P_{34}^2) \\ &= KK'. \end{aligned} \tag{\gamma}$$

8. If, now, a general rotation, $\mathfrak{S}'' \neq 0$, be given, we may decompose this into two simple rotations with fixed planes absolutely at right angles to each other by the aid of the above formulae (γ).

Squaring and adding the equations (γ) we have

$$\begin{aligned} a''^2 + b''^2 + c''^2 + f''^2 + g''^2 + h''^2 &= K^2(P_{12}^2 + P_{13}^2 + P_{14}^2 + P_{23}^2 + P_{24}^2 + P_{34}^2) \\ &\quad + 2KK'(P_{12}P_{34} + P_{13}P_{42} + P_{14}P_{23}) \\ &\quad + K'^2(P_{12}^2 + P_{13}^2 + P_{14}^2 + P_{23}^2 + P_{24}^2 + P_{34}^2) \\ &= K^2 + K'^2. \end{aligned}$$

From this equation, in combination with the equation $\mathfrak{S}'' = KK'$, we can determine K and K' . These equations have four pairs of solutions. Any one of these being given, the others are deduced from it, (1), by interchanging the values of K and K' , (2) by changing the signs of the values, and (3) by both interchanging the values and changing the signs.

Again, from the equation (γ), we obtain at once

$$\begin{aligned} P_{12} &= \frac{Ka'' - K'f''}{K^2 - K'^2} & P_{34} &= \frac{Kf'' - K'a''}{K^2 - K'^2}, \\ P_{13} &= \frac{Kb'' - K'g''}{K^2 - K'^2} & P_{42} &= \frac{Kg'' - K'b''}{K^2 - K'^2}, \\ P_{14} &= \frac{Kc'' - K'h''}{K^2 - K'^2} & P_{23} &= \frac{Kh'' - K'c''}{K^2 - K'^2}. \end{aligned}$$

From these equations it appears that if the signs of K and K' be changed, those of the P_{ik} are changed at the same time. In other words, such a change of signs does not affect the position of the fixed plane, but merely changes the point from which it is viewed from one side of the plane to the other.

Again, an interchange of K and K' converts the P_{ik} of the plane into those of the absolutely perpendicular plane.

Our system of solutions leads therefore to only one pair of fixed planes. We have accordingly the following proposition:

Every rotation of a four-dimensional space for which $S \neq 0$ can be reduced to a succession of two simple rotations whose fixed planes are absolutely perpendicular to each other. This decomposition can be effected in only one way. The quantity θ is the product of the tangents of half the angles of the simple rotations. The quantity

$$B = 1 + a^2 + b^2 + c^2 + f^2 + g^2 + h^2 + S^2 \\ = (1 + K^2 + K'^2 + K^2 K'^2) = (1 + K^2)(1 + K'^2),$$

i. e. it is the product of the squares of the secants of the two angles of the two simple rotations.

If we choose as axes of coordinates four lines lying two (x and y) in the fixed plane of the one simple component rotation, and two (z and w) in the fixed plane of the other, the equations of the rotation reduce evidently to the form

$$\begin{aligned} x' &= x \cos \phi - y \sin \phi, & z' &= z \cos \phi' - w \sin \phi', \\ y' &= x \sin \phi + y \cos \phi, & w' &= z \sin \phi' + w \cos \phi'. \end{aligned} \quad (\delta)$$

The simple rotations occur when one of the angles ϕ, ϕ' is 0. In conformity with the use of the name simple rotation, the general rotation may be called a double rotation.

9. In closing, it remains to be noted that the orthogonal transformations defined by equations (2) and (3) include not only the ∞^6 rotations of the four-dimensional space about the origin, but also an equal number of other transformations which are most simply described as combinations of the preceding rotations with a *reflection* on any lineoid through the origin. For convenience we will call this second class of transformations the *conjugate* transformations.

For example, the transformation $x' = -x, y' = y, z' = z, w' = w$ evidently belongs to the orthogonal system, but is no rotation. It is a reflection on the lineoid $x = 0$. This reflection leaves all points in the planes of xy, xz , and xw fixed, while the planes of zw, yw , and yz are reflected on the axis of y .

A simple algebraic criterion serves to distinguish between the rotations and the conjugate transformations. The square of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

of any orthogonal transformation is, as is well known, equal to $+1$, and consequently the determinant itself is equal to either $+1$ or -1 .

Those transformations whose determinant is $+1$ are rotations of the four-dimensional space about the origin.

If those transformations whose determinants are -1 be followed by a reflection on any lineoid through the origin, say by the reflection $x' = -x$, $y' = y$, $z' = z$, $w' = w$, the determinant of the resultant transformation is

$$\begin{vmatrix} -a_1 & b_1 & c_1 & d_1 \\ -a_2 & b_2 & c_2 & d_2 \\ -a_3 & b_3 & c_3 & d_3 \\ -a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

and is therefore $+1$. The resultant transformation is accordingly a rotation.

It appears, therefore, that all orthogonal transformations of determinant -1 are conjugate transformations.

Since the determinant of the combination of two linear transformations is the product of the determinants of the two component transformations, it follows that the combination of two rotations, or of two conjugate transformations, is a rotation, while the combination of a rotation with a conjugate transformation is a conjugate transformation.

The rotations accordingly form a *group*, while the conjugate transformations do not.

The conjugate transformations do not in general leave any point except the origin fixed. They may, however, leave a plane, and, in the particular case of reflections, a lineoid fixed.

The further treatment of this subject I reserve for another paper on groups of rotations in four-dimensional space, to which the present article is intended largely as a preface.